CONSTRUCTION OF THE TRANSREAL NUMBERS

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July 28, 2014

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Abstract

The transreal numbers, proposed by James Anderson, are an extension of the real numbers. This new set is closed under the four arithmetical operations: addition, subtraction, multiplication and division. In particular, division by zero is allowed. Anderson introduced the transreals intuitively and axiomatically. In this paper we propose a construction of the transreals from the reals. Thus the transreal numbers and their arithmetic arise as consequences of real numbers. We define the set of transreal numbers as a certain class of subsets of ordered pairs of real numbers and we show that, in an appropriate sense, there is a copy of the real numbers in this new set.

2010 Mathematics Subject Classification:03H15 (primary); 11A99 (secondary)

1 Introduction

The impossibility of division by zero, in the real numbers, is well known. One of the difficulties in defining such division is that both historical and currently popular interpretations of the division operation are not valid when the divisor is zero. For example the integral equality $n/d = m$ can be interpreted as follows: n objects can be divided into (set out as) d groups of m objects. This account of division makes no sense, for non-zero n, when d is zero because there is no number such that zero groups ($d = 0$), of any m objects, sum to $n \neq 0$. Even if we dispense with historical and pedagogical models, by operating formally, a problem remains. Division in the real numbers, $\mathbb{R}$, is multiplication by the multiplicative inverse. That is, if $a, b \in \mathbb{R}$ and $b \neq 0$ then a/b means $a \times b^{-1}$, where b^{-1} is a real number such that $b \times b^{-1} = 1$. Now if we wish to allow a denominator of zero, we must have a multiplicative inverse of zero. This is not possible in the usual definition of multiplication because, if there is $c \in \mathbb{R}$ such that $0 \times c = 1$, we would have $0 = 0 \times c = 1$, which is absurd! That said, it is clear that if we want to divide by zero, we need to extend the definition of division and, perhaps, the definition of number.

In the 2000s James Anderson[1] proposed the set of *transreal numbers* [3], a set where there are fractions with a denominator of zero. Anderson applies this theory to computer programming. Transreal arithmetic avoids exceptions and machine halts that would otherwise occur when a program instructs a division by zero. James Anderson posits, in addition to the real numbers, the existence of three new numbers: $1/0$, $-1/0$ and $0/0$, respectively called *infinity, negative infinity* and *nullity*. He calls the set of real numbers, together with these three new elements, the set of *transreal numbers* and defines ordering and a convenient arithmetic in this new set of numbers. In [2] Anderson considers the syntactic application of the rules for adding and multiplying fractions, notwithstanding the fact that fractions may have a zero denominator. That is he analyses what arithmetic is generated when the rules $\dfrac{x}{y} + \dfrac{w}{z} = \dfrac{xz + wy}{yz}$ and $\dfrac{x}{y} \times \dfrac{w}{z} = \dfrac{xw}{yz}$ are applied to fractions which may have a zero denominator. In [4] he proposes the set of transrational numbers, $\mathbb{Q}^T := \mathbb{Q} \cup \{-1/0, 1/0, 0/0\}$. Next the transrel numbers are introduced by listing their axioms [7]. In [5] Anderson extends the trigonometric, logarithmic and exponential functions to the transreal numbers and, in [6], he proposes a topology for transreal space and establishes the transmetric.

The set of transreals and transreal arithmetic are established by the axioms published in [7]. By contrast we propose a construction of the transreals from the reals. Thus the transreal numbers and their arithmetic arise as a consequence of the reals and not by free-standing axioms. It is important to emphasise that this text is not meant to show applications of the transreal numbers, rather the aim is to provide a mathematical substantiation for this new number system.

2 Initial Considerations

We believe that the transreal numbers are going through a common process in the history of mathematics. The real numbers themselves were initially conceived intuitively. Positive integers and positive rationals are present in the earliest records of mathematics but the recognition of irrational numbers is attributed to the Greeks of the fourth century BC [13]. Over time the real numbers were widely used and were informally understood to be in a bijective correspondence with the set of points on a straight line. Despite this understanding, for many people the irrational numbers were not accepted as numbers, but as convenient objects in certain studies [13]. The advent of the differential and integral calculus, around seventeenth century, brought new ideas and, together with these new ideas, controversies about their methods. These controversies were partially responsible for causing a move toward the formalisation of the mathematical concepts of number, in other words, the establishment of numbers without the assumption of geometrical intuition. Already in the eighteenth century, efforts were made to formalise the real numbers, but their consolidation occurred only in the nineteenth century with a construction from the rational numbers by Dedekind. Dedekind's motivation was to establish the set of real numbers, not just by the admission of its existence, but by constructing the real numbers from numbers that were already established. Another example of this process occurred with the complex numbers. When, in the sixteenth century, Bombelli found the square root of a negative number, while solving an equation of the third degree, he had the courage to operate on this object by assuming that it followed the arithmetical properties of the real numbers. He found, indeed, at the end of his calculation, a solution to the equation in question. At that moment, Bombelli was not preoccupied with rigour nor with the interpretation of the strange object, he was just brave and supposed the existence of new

[1] James A. D. W. Anderson is currently a teacher and research at the School of Systems Engineering, University of Reading, England.

entities that could be called numbers. Complex numbers were studied by other mathematicians but with the phenomenological quality of being *imaginary* numbers. Only in the nineteenth century, did Hamilton give a rigorous definition of the complex numbers and their arithmetic, deducing their properties from the properties of real numbers. The odyssey of the hyperreal numbers should also be mentioned. One of the fundamental concepts of the differential and integral calculus is now understood to be the limit. But Leibniz, one of the founders of modern calculus, did not use the idea of a limit. For instance he did not take the limit of a number tending to zero. Leibniz took a fixed number that was infinitely close to zero [10]. Even without a rigorous definition of infinitesimal numbers, formalising the idea of infinitely small numbers, Leibniz was still able to deduced several results of modern calculus. The infinitesimals suffered severe criticism and only in the 1960s did Robinson construct the infinitesimal numbers from the real numbers and deduce the properties foreseen by Leibniz.

Just as the aforementioned numbers were introduced intuitively, so the transreals were initially proposed, by James Anderson, using intuitions backed by an appeal to geometry. The concept of infinity and negative infinity, augmenting the set of real numbers, was already well known. "In integration theory it is frequently convenient to adjoin the two symbols $-\infty$, $+\infty$ to the real number system $\mathbb{R}$. (It is stressed that these symbols are not real numbers.)" [8]. So infinities were widely accepted as useful but were not, in the context of calculus, considered to be numbers. Infinity has long intrigued the human mind. The first to make a systematic study of infinity was Georg Cantor [9]. Cantor introduced a naive set theory. In Cantor's theory infinity, so common in mathematics, is a proper object of set theory and not a number to be joined to the real numbers. Nullity, in turn, was idealised by Anderson in projective geometry. A model for this geometry is to define each point in the projective plane as a certain class of points in $\mathbb{R}^3 \setminus \{(0,0,0)\}$ so that the point $(0,0,0)$ is not part of the system. An important feature of this model is the fact that all triples of the form (x,y,z), with $z \neq 0$, are equivalent to $(x/z, y/z, 1)$. Which would not make sense if $z = 0$. Further, in an appropriate way, the classes of points of the form (x,y,z), with $z = 0$ and x, y not simultaneously zero, are also considered as points. Operating directly on points of the form $(x,y,0)$, in the model of projective geometry, avoids introducing the undefined fractions $(x/0, y/0, 0/0)$. The points of this type, $(x,y,0)$, with $x \neq 0$ or $y \neq 0$, are called *ideal points*. In the projective plane, any two parallel lines (in the Euclidean sense) intersect (in the projective sense) in an ideal point. For this reason the ideal points are also called *points at infinity* [12]. Anderson notes that $(0,0,0)$ is not considered. For him the inclusion of the point $(0,0,0)$, in the model of projective geometry, brings several advantages to computing, especially in controlling robots which need to understand the shape and arrangement of objects in space and how they change over time. Anderson defends this thesis and refers to the point $(0,0,0)$ as the *point at nullity* [1] .

3 Construction of the transreal numbers from the real numbers

In what follows, we propose a construction of the transreal numbers from the real numbers. We define, for a given class of subsets of pairs of the real numbers, arithmetic operations (using real arithmetic operations) and we show that there is a copy of the real numbers in this class. Thus the transreal numbers and the arithmetic proposed by James Anderson become consequences of these definitions and of the properties of real numbers.

The idea presented here is based on the concept of equivalence between fractions of integers.

It is well known that if $x, y, w, z \in \mathbb{Z}$, with $y > 0$ and $z > 0$ then the fractions x/y and w/z are equivalent if and only if $xz = wy$. Incidentally this is the principle used in the construction of the rational numbers, from the integers, where the equivalence relation $(x, y) \sim (w, z)$, defined on $\mathbb{Z} \times \mathbb{Z}^+$, holds if and only if $xz = wy$ [11]. Note that according to this definition, the relation $\sim$ is not an equivalence relation if we allow the second element to be zero. Indeed we would have, for example, $(1, 2) \sim (0, 0)$ and $(0, 0) \sim (1, 3)$ but we would not have the transitive case $(1, 2) \sim (1, 3)$. We use an equivalence relation to construct the transreal numbers, however we adapt the definition, of the relation, to allow the second element to be zero.

Definition 1. Let $T := \big\{(x, y);\ x, y \in \mathbb{R} \text{ and } y \geq 0\big\}$. Given $(x, y), (w, z) \in T$, we say that $(x, y) \sim (w, z)$, that is (x, y) is equivalent to (w, z), with respect to $\sim$, if and only if there is a positive $\alpha \in \mathbb{R}$ such that $x = \alpha w$ and $y = \alpha z$.

Proposition 2. The relation $\sim$ is an equivalence relation on T.

Proof. The Reflexive property of $\sim$ is immediate. Now let $(x, y), (w, z), (u, v) \in T$ such that $(x, y) \sim (w, z)$ and $(w, z) \sim (u, v)$. Then there are positive $\alpha, \beta \in \mathbb{R}$ such that $x = \alpha w$, $y = \alpha z$, $w = \beta u$ and $z = \beta v$. The symmetric property follows from $w = \dfrac{1}{\alpha}x$ and $z = \dfrac{1}{\alpha}y$. The transitive property follows from $x = \alpha\beta u$ and $y = \alpha\beta v$. □

For each $(x, y) \in T$, let us denote by $[x, y]$ the equivalence class of (x, y), that is $[x, y] := \big\{(w, z) \in T;\ (w, z) \sim (x, y)\big\}$. Let us denote by $T/\!\sim$ the quotient set of T with respect to $\sim$, that is $T/\!\sim := \big\{[x, y];\ (x, y) \in T\big\}$.

Proposition 3. It follows that $T/\!\sim\ = \big\{[t, 1];\ t \in \mathbb{R}\big\} \bigcup \big\{[0, 0], [1, 0], [-1, 0]\big\}$. Furthermore the elements $[t, 1]$, $[0, 0]$, $[1, 0]$, $[-1, 0]$ are pairwise distinct and for each $t, s \in \mathbb{R}$, it is the case that $[t, 1] \neq [s, 1]$ whenever $t \neq s$.

Proof. If $[x, y] \in T/\!\sim$ then either $y > 0$ or $y = 0$. If $y > 0$ then $[x, y] = [x/y, 1] \in \big\{[t, 1];\ t \in \mathbb{R}\big\}$ because $x = y\dfrac{x}{y}$ and $y = y \times 1$. On the other hand

$$y = 0 \Rightarrow \begin{cases} \text{if } x = 0 \text{ then } [x, y] = [0, 0], \\ \text{if } x > 0 \text{ then } [x, y] = [1, 0] \text{ because } x = x \times 1 \text{ and } y = x \times 0 \\ \text{if } x < 0 \text{ then } [x, y] = [-1, 0] \text{ because } x = -x \times (-1) \text{ and } y = -x \times 0 \end{cases}.$$

The rest of the proof follows immediately. □

Now let us define operations on $T/\!\sim$ which extend the arithmetical operations between real numbers.

Definition 4. Given $[x, y], [w, z] \in T/\!\sim$ let us define:

a) (Addition) $[x, y] \oplus [w, z] := \begin{cases} [2x, y] & , \text{ if } [x, y] = [w, z] \\ [xz + wy, yz] & , \text{ if } [x, y] \neq [w, z] \end{cases}$,

b) (Multiplication) $[x, y] \otimes [w, z] := [xw, yz]$,

c) (Opposite) $\ominus[x, y] := [-x, y]$,

d) (Reciprocal) $[x,y]^{(-1)} := \begin{cases} [y,x] & , \text{ if } x \geq 0 \\ [-y,-x] & , \text{ if } x < 0 \end{cases}$,

e) (Subtraction) $[x,y] \ominus [w,z] := [x,y] \oplus (\ominus[w,z])$ and

f) (Division) $[x,y] \oslash [w,z] := [x,y] \otimes [w,z]^{(-1)}$.

Proposition 5. The operations $\oplus$, $\otimes$, $\ominus$ and $^{(-1)}$ are well defined. That is $[x,y] \oplus [w,z]$, $[x,y] \otimes [w,z]$, $\ominus[x,y]$ and $[x,y]^{(-1)}$ are independent of the choice of representatives of the classes $[x,y]$ and $[w,z]$.

Proof. Let $[x,y],[w,z] \in T/\sim$, $(x',y') \in [x,y]$ and $(w',z') \in [w,z]$. Then there are positive $\alpha, \beta \in \mathbb{R}$ such that $x = \alpha x'$, $y = \alpha y'$, $w = \beta w'$ and $z = \beta z'$.

a) If $[x,y] = [w,z]$ then $[x',y'] = [w',z']$. Thus $[x,y] \oplus [w,z] = [2x,y] = [2x',y'] = [x',y'] \oplus [w',z']$. If $[x,y] \neq [w,z]$ then $[x',y'] \neq [w',z']$ and $xz + wy = \alpha x' \beta z' + \beta w' \alpha y' = \alpha\beta(x'z' + w'y')$ and $yz = \alpha y' \beta z' = \alpha\beta y'z'$. Thus $[x,y] \oplus [w,z] = [xz+wy, yz] = [x'z'+w'y', y'z'] = [x',y'] \oplus [w',z']$.

b) Notice that $xw = \alpha x' \beta w' = \alpha\beta x'w'$ and $yz = \alpha y' \beta z' = \alpha\beta y'z'$, whence $[x,y] \otimes [w,z] = [xw, yz] = [x'w', y'z'] = [x',y'] \otimes [w',z']$.

c) Note that $-x = -(\alpha x') = \alpha(-x')$ and $y = \alpha y'$. Thus $\ominus[x,y] = [-x,y] = [-x',y'] = \ominus[x',y']$.

d) Notice that $y = \alpha y'$, $x = \alpha x'$, $-y = \alpha(-y')$ and $-x = \alpha(-x')$. Thus if $x \geq 0$ then $[x,y]^{(-1)} = [y,x] = [y',x'] = [x',y']^{(-1)}$ and if $x < 0$ then $[x,y]^{(-1)} = [-y,-x] = [-y',-x'] = [x',y']^{(-1)}$.

□

Now let us define an order relation on $T/\sim$.

Definition 6. Let arbitrary $[x,y],[w,z] \in T/\sim$. We say that $[x,y] \prec [w,z]$ if and only if either $[x,y] = [-1,0]$ and $[w,z] = [1,0]$ or else $xz < wy$. Furthermore we say that $[x,y] \preceq [w,z]$ if and only if $[x,y] \prec [w,z]$ or $[x,y] = [w,z]$.

Notice that the relation $\preceq$ is well defined and is an order relation on $T/\sim$.
The following theorem assures us that, in an appropriate sense, $\mathbb{R}$ is subset of $T/\sim$.

Theorem 7. The set $R := \{[t,1];\ t \in \mathbb{R}\}$ is a complete ordered field.

Proof. The result follows from the fact that $\pi : \mathbb{R} \longrightarrow R$, $\pi(t) = [t,1]$ is bijective and, for any $t,s \in \mathbb{R}$,

i) $\pi(t) \oplus \pi(s) = \pi(t+s)$,

ii) $\pi(t) \otimes \pi(s) = \pi(ts)$ and

iii) $\pi(t) \preceq \pi(s)$ if and only if $t \leq s$,

and from the fact that $\mathbb{R}$ is a complete ordered field. □

Note that for each $t \in \mathbb{R}$, $\ominus[t,1] = [-t,1]$ and if $t \neq 0$ then $[t,1]^{(-1)} = [t^{-1},1]$.

Observation 8. Since π is an isomorphism of complete ordered fields between R and $\mathbb{R}$, we can say that R is a "copy" of $\mathbb{R}$ in $T/\sim$. Therefore let us abuse language and notation: henceforth R will be denoted by $\mathbb{R}$ and will be called the set of real numbers and each $[t,1] \in R$ will be denoted, simply, by t and will be called a real number. In this sense we can say that $\mathbb{R} \subset T/\sim$ and we replace the symbols $\oplus$, $\otimes$, $\ominus$, $\oslash$, $^{(-1)}$, $\prec$ and $\preceq$, respectively, by $+$, $\times$, $-$, $/$, $^{-1}$, $<$ and $\leq$.

Let us define and denote *negative infinity, infinity* and *nullity*, respectively, by $-\infty := [-1,0]$, $\infty := [1,0]$ and $\Phi := [0,0]$. Let us refer to the elements of $T/\sim$ as *transreal numbers*, thus $T/\sim$ will be the *set of transreal numbers*. Let us denote $\mathbb{R}^T := T/\sim$. Whence $\mathbb{R}^T = \mathbb{R} \cup \{-\infty, \infty, \Phi\}$. Let us refer to the elements $-\infty, \infty$ and Φ as *strictly transreal numbers.*

The next theorem sets out transreal arithmetic and ordering.

Theorem 9. For each $x \in \mathbb{R}^T$, it follows that:

a) $-\Phi = \Phi, \quad -(\infty) = -\infty \quad$ and $\quad -(-\infty) = \infty,$

b) $0^{-1} = \infty, \quad \Phi^{-1} = \Phi, \quad (-\infty)^{-1} = 0 \quad$ and $\quad \infty^{-1} = 0,$

c) $\Phi + x = \Phi, \quad -\infty + x = \begin{cases} \Phi & \text{, if } x \in \{\infty, \Phi\} \\ -\infty & \text{, otherwise} \end{cases}$ and $\infty + x = \begin{cases} \Phi & \text{, if } x \in \{-\infty, \Phi\} \\ \infty & \text{, otherwise} \end{cases}$
and

d) $\Phi \times x = \Phi, \quad -\infty \times x = \begin{cases} \Phi & \text{, if } x \in \{0, \Phi\} \\ \infty & \text{, if } x < 0 \\ -\infty & \text{, if } x > 0 \end{cases}$ and $\infty \times x = \begin{cases} \Phi & \text{, if } x \in \{0, \Phi\} \\ -\infty & \text{, if } x < 0 \\ \infty & \text{, if } x > 0 \end{cases}$.

e) If $x \in \mathbb{R}$ then $-\infty < x < \infty$.

f) The following does not hold $x < \Phi$ or $\Phi < x$.

Proof. Denote $x = [x_1, x_2]$.

a) $-\Phi = -[0,0] = [0,0] = \Phi,$
$-(\infty) = -[1,0] = [-1,0] = -\infty$ and
$-(-\infty) = -[-1,0] = [1,0] = \infty.$

b) $0^{-1} = [0,1]^{-1} = [1,0] = \infty,$
$\Phi^{-1} = [0,0]^{-1} = [0,0] = \Phi,$
$(-\infty)^{-1} = [-1,0]^{-1} = [-0,-(-1)] = [0,1] = 0$ and
$\infty^{-1} = [1,0]^{-1} = [0,1] = 0.$

c) $\Phi + x = [0,0] + [x_1,x_2] = [0 \times x_2 + x_1 \times 0, 0 \times x_2] = [0,0] = \Phi,$
$\infty + (-\infty) = [1,0] + [-1,0] = [1 \times 0 + (-1) \times 0, 0 \times 0] = [0,0] = \Phi,$
$\infty + \Phi = [1,0] + [0,0] = [1 \times 0 + 0 \times 0, 0 \times 0] = [0,0] = \Phi$ and
$\infty + \infty = [1,0] + [1,0] = [2,0] = [1,0] = \infty.$
If $x \in \mathbb{R}$, $\infty + x = [1,0] + [x,1] = [1 \times 1 + x \times 0, 0 \times 1] = [1,0] = \infty.$
The addition $-\infty + x$ holds analogously.

d) $\Phi \times x = [0, 0] \times [x_1, x_2] = [0 \times x_1, 0 \times x_2] = [0, 0] = \Phi$,
$\infty \times 0 = [1, 0] \times [0, 1] = [1 \times 0, 0 \times 1] = [0, 0] = \Phi$ and
$\infty \times \Phi = [1, 0] \times [0, 0] = [1 \times 0, 0 \times 0] = [0, 0] = \Phi$.
If $x < 0$ then $x_1 < 0$, whence $\infty \times x = [1, 0] \times [x_1, x_2] = [1 \times x_1, 0 \times x_2] = [x_1, 0] = [-1, 0] = -\infty$.
If $x > 0$ then $x_1 > 0$, whence $\infty \times x = [1, 0] \times [x_1, x_2] = [1 \times x_1, 0 \times x_2] = [x_1, 0] = [1, 0] = \infty$.
The multiplication $-\infty \times x$ holds analogously.

e) If $x = [x_1, x_2] \in \mathbb{R}$ then $x_2 > 0$, whence $-1 \times x_2 = -x_2 < 0 = x_1 \times 0$ and $x_1 \times 0 = 0 < x_2 = 1 \times x_2$.

f) $\Phi \neq [-1, 0]$, $\Phi \neq [1, 0]$ and $x_1 \times 0 = 0 \not< 0 = 0 \times x_2$.

$\square$

Corollary 10. Let $x, y \in \mathbb{R}$ where $x > 0$ and $y < 0$. It follows that:

a) $\dfrac{x}{0} = \infty$,

b) $\dfrac{y}{0} = -\infty$ and

c) $\dfrac{0}{0} = \Phi$.

Proof. a) $\dfrac{x}{0} = x \times 0^{-1} = x \times \infty = \infty$,

b) $\dfrac{y}{0} = y \times 0^{-1} = y \times \infty = -\infty$ and

c) $\dfrac{0}{0} = 0 \times 0^{-1} = 0 \times \infty = \Phi$.

$\square$

In the next theorem we establish on $\mathbb{R}^T$ some arithmetical and ordering properties that are true on $\mathbb{R}$. Regarding the properties that are not true for all transreal numbers, we indicate the necessary restrictions.

Theorem 11. Let $x, y, z \in \mathbb{R}^T$. It follows that:

a) (Additive Commutativity) $x + y = y + x$,

b) (Additive Associativity) $(x + y) + z = x + (y + z)$,

c) (Additive Identity) $x + 0 = 0 + x = 0$,

d) (Additive Inverse) if $x \notin \{-\infty, \infty, \Phi\}$ then $x - x = 0$,

e) (Multiplicative Commutativity) $x \times y = y \times x$,

f) (Multiplicative Associativity) $(x \times y) \times z = x \times (y \times z)$,

g) (Multiplicative Identity) $x \times 1 = 1 \times x = x$,

h) (Multiplicative Inverse) if $x \notin \{0, -\infty, \infty, \Phi\}$ then $\dfrac{x}{x} = 1$,

i) (Distributivity) if $x \notin \{-\infty, \infty\}$ or $yz > 0$ or $y+z = 0$ or $x, y, z \in \{-\infty, \infty\}$ then $x \times (y+z) = (x \times y) + (x \times z)$ and $(y + z) \times x = (y \times x) + (z \times x)$,

j) (Additive Monotonicity) if not simultaneously $z = -\infty$, $x = -\infty$ and $y = \infty$ and not simultaneously $z = -\infty$, $x \in \mathbb{R}$ and $y = \infty$ and not simultaneously $z = \infty$, $x = -\infty$ and $y = \infty$ and not simultaneously $z = \infty$, $x = -\infty$ and $y \in \mathbb{R}$ then

$$x \leq y \Rightarrow x + z \leq y + z,$$

k) (Multiplicative Monotonicity) if not simultaneously $z = 0$, $x = -\infty$ and $y \in \mathbb{R}$ and not simultaneously $z = 0$, $x \in \mathbb{R}$ and $y = \infty$ then

$$x \leq y \ \text{ and } \ z \geq 0 \Rightarrow xz \leq yz \ \text{ and}$$

l) (Existence of Supremum) if $A \subset \mathbb{R}^T \setminus \{\Phi\}$ is non-empty then A has supremum in $\mathbb{R}^T$.

Notice that, as show in the following examples, the restrictions on the items (d), (h), (i), (j) and (k) of the Theorem 11 are indeed necessary.

Example 12. From Theorem 9, $\Phi - \Phi = -\infty - (-\infty) = \infty - \infty = \Phi$.

Example 13. From Theorem 9, $\dfrac{0}{0} = \dfrac{\Phi}{\Phi} = \dfrac{-\infty}{-\infty} = \dfrac{\infty}{\infty} = \Phi$.

Example 14. $\infty \times (-2 + 3) = \infty \times 1 = \infty \neq \Phi = -\infty + \infty = (\infty \times (-2)) + (\infty \times 3)$.

$\infty \times (0 + 3) = \infty \times 3 = \infty \neq \Phi = \Phi + \infty = (\infty \times 0) + (\infty \times 3)$.

$\infty \times (-\infty + 3) = \infty \times (-\infty) = -\infty \neq \Phi = -\infty + \infty = (\infty \times (-\infty)) + (\infty \times 3)$.

Example 15. $-\infty \leq \infty$ and $-\infty + (-\infty) = -\infty \nleq \Phi = \infty + (-\infty)$.

If $x \in \mathbb{R}$ then $x \leq \infty$ and $x + (-\infty) = -\infty \nleq \Phi = \infty + (-\infty)$.

$-\infty \leq \infty$ and $-\infty + \infty = \Phi \nleq \infty = \infty + \infty$.

If $y \in \mathbb{R}$ then $-\infty \leq y$ and $-\infty + \infty = \Phi \nleq \infty = y + \infty$.

Example 16. If $y \in \mathbb{R}$ then $-\infty \leq y$ and $-\infty \times 0 = \Phi \nleq 0 = y \times 0$.

If $x \in \mathbb{R}$ then $x \leq \infty$ and $x \times 0 = 0 \nleq \Phi = \infty \times 0$.

Although somewhat tedious, it is necessary to prove the previous theorem.

Note that if it is not the case that $[x, y] = [w, z] = [-1, 0]$ nor $[x, y] = [w, z] = [1, 0]$ then $[x, y] + [w, z] = [xz + wy, yz]$, even though $[x, y] = [w, z]$. Furthermore we recall that the *real sign function* is defined by

$$\mathrm{sgn} : \mathbb{R} \longrightarrow \mathbb{R},\ \mathrm{sgn}(x) = \begin{cases} -1 & , \text{ if } x < 0 \\ 0 & , \text{ if } x = 0 \\ 1 & , \text{ if } x > 0 \end{cases} .$$

Notice that for all $x, y \in \mathbb{R}$ it is the case that $\mathrm{sgn}(x)\times\mathrm{sgn}(y) = \mathrm{sgn}(xy)$. Moreover $[x, 0] = [\mathrm{sgn}(x), 0]$. We use these observations in what follows.

Proof of Theorem 11. Let us denote $x = [x_1, x_2]$, $y = [y_1, y_2]$ and $z = [z_1, z_2]$.

a) If $x = y$ then the result is immediate. Otherwise $x + y = [x_1, x_2] + [y_1, y_2] = [x_1y_2 + y_1x_2, x_2y_2] = [y_1x_2 + x_1y_2, y_2x_2] = [y_1, y_2] + [x_1, x_2] = y + x$.

b) if $y = \Phi$ then

$x + (\Phi + z) = x + \Phi = \Phi = \Phi + z = (x + \Phi) + z$.

If $y = -\infty$ then

$\Phi + (-\infty + \Phi) = (-\infty + \Phi) + \Phi = (\Phi + (-\infty)) + \Phi$.

$\Phi + (-\infty + (-\infty)) = \Phi + (-\infty) = (\Phi + (-\infty)) + (-\infty)$.

$\Phi + (-\infty + \infty) = \Phi + \Phi = \Phi = \Phi + \infty = (\Phi + (-\infty)) + \infty$.

$\Phi + (-\infty + z) = \Phi + (-\infty) = \Phi = \Phi + z = (\Phi + (-\infty)) + z$, for all $z \in \mathbb{R}$.

$-\infty + (-\infty + \Phi) = -\infty + \Phi = (-\infty + (-\infty)) + \Phi$.

$-\infty + (-\infty + (-\infty)) = -\infty = (-\infty + (-\infty)) + (-\infty)$.

$-\infty + (-\infty + \infty) = -\infty + \Phi = \Phi = -\infty + \infty = (-\infty + (-\infty)) + \infty$.

$-\infty + (-\infty + z) = -\infty + (-\infty) = -\infty = -\infty + z = (-\infty + (-\infty)) + z$, for all $z \in \mathbb{R}$.

$\infty + (-\infty + \Phi) = \infty + \Phi = \Phi = \Phi + \Phi = (\infty + (-\infty)) + \Phi$.

$\infty + (-\infty + (-\infty)) = \infty + (-\infty) = \Phi = \Phi + (-\infty) = (\infty + (-\infty)) + (-\infty)$.

$\infty + (-\infty + \infty) = \infty + \Phi = \Phi + \infty = (\infty + (-\infty)) + \infty$.

$\infty + (-\infty + z) = \infty + (-\infty) = \Phi = \Phi + z = (\infty + (-\infty)) + z$, for all $z \in \mathbb{R}$.

$x + (-\infty + \Phi) = x + \Phi = \Phi = -\infty + \Phi = (x + (-\infty)) + \Phi$, for all $x \in \mathbb{R}$.

$x + (-\infty + (-\infty)) = x + (-\infty) = -\infty = -\infty + (-\infty) = (x + (-\infty)) + (-\infty)$, for all $x \in \mathbb{R}$.

$x + (-\infty + \infty) = x + \Phi = \Phi = -\infty + \infty = (x + (-\infty)) + \infty$, for all $x \in \mathbb{R}$.

$x + (-\infty + z) = x + (-\infty) = -\infty = -\infty + z = (x + (-\infty)) + z$, for all $z \in \mathbb{R}$ and all $x \in \mathbb{R}$.

If $y = \infty$ the result holds analogously.

If $y \in \mathbb{R}$ then

$\Phi + (y + \Phi) = (y + \Phi) + \Phi = (\Phi + y) + \Phi$.

$\Phi + (y + (-\infty)) = \Phi + (-\infty) = (\Phi + y) + (-\infty)$.

$\Phi + (y + \infty) = \Phi + \infty = (\Phi + y) + \infty$.

$\Phi + (y + z) = \Phi = \Phi + z = (\Phi + y) + z$, for all $z \in \mathbb{R}$.

$-\infty + (y + \Phi) = -\infty + \Phi = (-\infty + y) + \Phi$.

$-\infty + (y + (-\infty)) = -\infty + (-\infty) = (-\infty + y) + (-\infty)$.

$-\infty + (y + \infty) = -\infty + \infty = (-\infty + y) + \infty$.

$-\infty + (y + z) = -\infty = -\infty + z = (-\infty + y) + z$, for all $z \in \mathbb{R}$.

$\infty + (y + \Phi) = \infty + \Phi = (\infty + y) + \Phi$.

$\infty + (y + (-\infty)) = \infty + (-\infty) = (\infty + y) + (-\infty)$.

$\infty + (y + \infty) = \infty + \infty = (\infty + y) + \infty$.

$\infty + (y + z) = \infty = \infty + z = (\infty + y) + z$, for all $z \in \mathbb{R}$.

$x + (y + \Phi) = x + \Phi = \Phi = (x + y) + \Phi$, for all $x \in \mathbb{R}$.

$x + (y + (-\infty)) = x + (-\infty) = -\infty = (x + y) + (-\infty)$, for all $x \in \mathbb{R}$.

$x + (y + \infty) = x + \infty = \infty = (x + y) + \infty$, for all $x \in \mathbb{R}$.

$x + (y + z) = (x + y) + z$, for all $z \in \mathbb{R}$ and for all $x \in \mathbb{R}$, from the additive associativity of real numbers.

c) $x + 0 = [x_1, x_2] + [0, 1] = [x_1 \times 1 + 0 \times x_2, x_2 \times 1] = [x_1, x_2] = x$.

d) This case is immediate.

e) $x \times y = [x_1, x_2] \times [y_1, y_2] = [x_1y_1, x_2y_2] = [y_1x_1, y_2x_2] = [y_1, y_2] \times [x_1, x_2] = y \times x$.

f) $(x \times y) \times z = ([x_1, x_2] \times [y_1, y_2]) \times [z_1, z_2] = ([x_1y_1, x_2y_2]) \times [z_1, z_2] = [(x_1y_1)z_1, (x_2y_2)z_2] = [x_1(y_1z_1), x_2(y_2z_2)] = [x_1, x_2] \times [y_1z_1, y_2z_2] = [x_1, x_2] \times ([y_1, y_2] \times [z_1, z_2]) = x \times (y \times z)$.

g) $x \times 1 = [x_1, x_2] \times [1, 1] = [x_1 \times 1, x_2 \times 1] = [x_1, x_2] = x$.

h) This case is immediate.

i) (I) $x \notin \{-\infty, \infty\}$.

Suppose $x = \Phi$. Then $x \times (y + z) = \Phi \times (y + z) = \Phi = \Phi + \Phi = (\Phi \times y) + (\Phi \times z) = (x \times y) + (x \times z)$.

Suppose $x \in \mathbb{R}$. If $y = z = \infty$ or $y = z = -\infty$ then $x \times (y + z) = x \times y = y = y + y = (x \times y) + (x \times y) = (x \times y) + (x \times z)$. Otherwise $x \times (y + z) = [x, 1] \times ([y_1, y_2] + [z_1, z_2]) = [x, 1] \times [y_1z_2 + z_1y_2, y_2z_2] = [x \times (y_1z_2 + z_1y_2), 1 \times (y_2z_2)] = [xy_1z_2 + xz_1y_2, y_2z_2] = [xy_1, y_2] + [xz_1, z_2] = ([x, 1] \times [y_1, y_2]) + ([x, 1] \times [z_1, z_2]) = (x \times y) + (x \times z)$.

(II) $yz > 0$. Note that $\mathrm{sgn}(y_1) = \mathrm{sgn}(z_1)$.

If $y = z = \infty$ or $y = z = -\infty$ then $x \times (y + z) = [x_1, x_2] \times ([y_1, 0] + [z_1, 0]) = [x_1, x_2] \times ([\mathrm{sgn}(y_1), 0] + [\mathrm{sgn}(z_1), 0]) = [x_1, x_2] \times ([\mathrm{sgn}(y_1), 0] + [\mathrm{sgn}(y_1), 0]) = [x_1, x_2] \times [\mathrm{sgn}(y_1), 0] = [x_1\mathrm{sgn}(y_1), x_2 \times 0] = [x_1\mathrm{sgn}(y_1), 0] = [x_1\mathrm{sgn}(y_1), 0] + [x_1\mathrm{sgn}(y_1), 0] = [x_1\mathrm{sgn}(y_1), 0] + [x_1\mathrm{sgn}(z_1), 0] = [x_1\mathrm{sgn}(y_1), x_2 \times 0] + [x_1\mathrm{sgn}(z_1), x_2 \times 0] = ([x_1, x_2] \times [\mathrm{sgn}(y_1), 0]) + ([x_1, x_2] \times [\mathrm{sgn}(z_1), 0]) = ([x_1, x_2] \times [y_1, 0]) + ([x_1, x_2] \times [z_1, 0]) = (x \times y) + (x \times z)$. Otherwise we have $x_2 = 0$ or

$x_2 > 0$. If $x_2 = 0$ then $x \times (y+z) = [x_1, 0] \times ([y_1, y_2] + [z_1, z_2]) = [x_1, 0] \times [y_1 z_2 + z_1 y_2, y_2 z_2] = [x_1 \times (y_1 z_2 + z_1 y_2), 0 \times (y_2 z_2)] = [x_1(y_1 z_2 + z_1 y_2), 0] = [\mathrm{sgn}(x_1)\mathrm{sgn}(y_1 z_2 + z_1 y_2), 0] = [\mathrm{sgn}(x_1)\mathrm{sgn}(y_1), 0]$ and $(x \times y) + (x \times z) = ([x_1, 0] \times [y_1, y_2]) + ([x_1, 0] \times [z_1, z_2]) = [x_1 y_1, 0 \times y_2] + [x_1 z_1, 0 \times z_2] = [x_1 y_1, 0] + [x_1 z_1, 0] = [\mathrm{sgn}(x_1)\mathrm{sgn}(y_1), 0] + [\mathrm{sgn}(x_1)\mathrm{sgn}(z_1), 0] = [\mathrm{sgn}(x_1)\mathrm{sgn}(y_1), 0] + [\mathrm{sgn}(x_1)\mathrm{sgn}(y_1), 0] = [\mathrm{sgn}(x_1)\mathrm{sgn}(y_1), 0]$. If $x_2 > 0$ then $x \times (y+z) = [x_1, x_2] \times ([y_1, y_2] + [z_1, z_2]) = [x_1, x_2] \times [y_1 z_2 + z_1 y_2, y_2 z_2] = [x_1 \times (y_1 z_2 + z_1 y_2), x_2 \times (y_2 z_2)] = [x_1(y_1 z_2 + z_1 y_2), x_2 y_2 z_2] = [x_2 x_1(y_1 z_2 + z_1 y_2), x_2(x_2 y_2 z_2)] = [x_1 y_1 x_2 z_2 + x_1 z_1 x_2 y_2, x_2 x_2 y_2 z_2] = [x_1 y_1, x_2 y_2] + [x_1 z_1, x_2 z_2] = ([x_1, x_2] \times [y_1, y_2]) + ([x_1, x_2] \times [z_1, z_2]) = (x \times y) + (x \times z)$.

(III) $y + z = 0$.

We have $[y_1 z_2 + z_1 y_2, y_2 z_2] = [y_1, y_2] + [z_1, z_2] = [0, 1]$. Thus $y_2 \neq 0$ and $z_2 \neq 0$, whence $y, z \in \mathbb{R}$ and $z = -y$. Thus $x \times (y+z) = x \times 0 = [x_1, x_2] \times [0, 1] = [x_1 \times 0, x_2 \times 1] = [0, x_2] = [0 \times x_2, x_2 x_2] = [0, x_2 x_2] = [x_1 y x_2 - x_1 y x_2, x_2 x_2] = [x_1 y, x_2] + [-x_1 y, x_2] = [x_1 y, x_2 \times 1] + [x_1(-y), x_2 \times 1] = ([x_1, x_2] \times [y, 1]) + ([x_1, x_2] \times [-y, 1]) = ([x_1, x_2] \times [y, 1]) + ([x_1, x_2] \times [z, 1]) = (x \times y) + (x \times z)$.

(IV) $x, y, z \in \{-\infty, \infty\}$.

If $y \neq z$ we can suppose, without loss of generality, that $x = \infty$, $y = -\infty$ and $z = \infty$, whence $x \times (y+z) = \infty \times (-\infty + \infty) = \infty \times \Phi = \Phi = -\infty + \infty = (\infty \times (-\infty)) + (\infty \times \infty) = (x \times y) + (x \times z)$. Otherwise we can suppose, without loss of generality, that $x = -\infty$, $y = \infty$ and $z = \infty$, whence $x \times (y+z) = -\infty \times (\infty + \infty) = -\infty \times \infty = -\infty = -\infty + (-\infty) = (-\infty \times \infty) + (-\infty \times \infty) = (x \times y) + (x \times z)$.

The equality $(y+z) \times x = (y \times x) + (z \times x)$ follows from the preceding equality and from multiplicative commutativity.

j) Suppose $x \leq y$. Then

$$\left\{\begin{array}{l} (\mathrm{a}_1)\ z = \Phi \text{ or} \\ (\mathrm{b}_1)\ z = -\infty \text{ or} \\ (\mathrm{c}_1)\ z = \infty \text{ or} \\ (\mathrm{d}_1)\ z \in \mathbb{R} \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} (\mathrm{a}_2)\ x = \Phi \text{ or} \\ (\mathrm{b}_2)\ x = -\infty \text{ or} \\ (\mathrm{c}_2)\ x = \infty \text{ or} \\ (\mathrm{d}_2)\ x \in \mathbb{R} \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} (\mathrm{a}_3)\ y = \Phi \text{ or} \\ (\mathrm{b}_3)\ y = -\infty \text{ or} \\ (\mathrm{c}_3)\ y = \infty \text{ or} \\ (\mathrm{d}_3)\ y \in \mathbb{R} \end{array}\right. .$$

Notice that the condition pairs (a$_2$) and (b$_3$), (a$_2$) and (c$_3$), (a$_2$) and (d$_3$), (b$_2$) and (a$_3$), (c$_2$) and (a$_3$), (d$_2$) and (a$_3$) do not occur because if $x = \Phi$ or $y = \Phi$ then $x = y = \Phi$. The pairs (c$_2$) and (b$_3$), (c$_2$) and (d$_3$) do not occur because if $x = \infty$ then $y = \infty$. Furthermore, the pairs (d$_2$) and (b$_3$) do not occur because if $y = -\infty$ then $x = -\infty$.

If (a$_1$) occurs then $x + z = x + \Phi = \Phi = y + \Phi = y + z$.

If (b$_1$) occurs then observe the following. If the pair (a$_2$) and (a$_3$) occurs then $x + z = \Phi + (-\infty) = y + z$. If the pair (b$_2$) and (b$_3$) occurs then $x + z = -\infty + (-\infty) = y + z$. By hypothesis the pair (b$_2$) and (c$_3$) does not occur. If the pair (b$_2$) and (d$_3$) occurs then $x + z = -\infty + (-\infty) = -\infty = y + (-\infty) = y + z$. If the pair (c$_2$) and (c$_3$) occurs then $x + z = \infty + (-\infty) = y + z$. By hypothesis the pair(d$_2$) and (c$_3$) does not occur. If the pair (d$_2$) and (d$_3$) occurs then $x + z = x + (-\infty) = -\infty = y + (-\infty) = y + z$.

If (c$_1$) occurs then the result follows analogously to the previous case.

If (d_1) occurs then observe the following. If the pair (a_2) and (a_3) occurs then $x+z=\Phi+z=y+z$. If the pair (b_2) and (b_3) occurs then $x+z=-\infty+z=y+z$. If the pair (b_2) and (c_3) occurs then $x+z=-\infty+z=-\infty<\infty=\infty+z=y+z$. If the pair ($b_2$) and ($d_3$) occurs then $x+z=-\infty+z=-\infty<y+z$. If the pair ($c_2$) and ($c_3$) occurs then $x+z=\infty+z=y+z$. If the pair (d_2) and (c_3) occurs then $x+z<\infty=\infty+z=y+z$. If the pair (d_2) and (d_3) occurs then the result follows from the real number order relation and real addition.

k) Suppose $x \leq y$. Then

$$\begin{cases} (a_1)\ z=\infty \text{ or} \\ (b_1)\ z\in\mathbb{R} \text{ with } z\geq 0 \end{cases} \quad \text{and} \quad \begin{cases} (a_2)\ x=\Phi \text{ or} \\ (b_2)\ x=-\infty \text{ or} \\ (c_2)\ x=\infty \text{ or} \\ (d_2)\ x\in\mathbb{R} \end{cases} \quad \text{and} \quad \begin{cases} (a_3)\ y=\Phi \text{ or} \\ (b_3)\ y=-\infty \text{ or} \\ (c_3)\ y=\infty \text{ or} \\ (d_3)\ y\in\mathbb{R} \end{cases}.$$

Notice that the condition pairs (a_2) and (b_3), (a_2) and (c_3), (a_2) and (d_3), (b_2) and (a_3), (c_2) and (a_3), (d_2) and (a_3) do not occur because if $x=\Phi$ or $y=\Phi$ then $x=y=\Phi$. Also the pairs (c_2) and (b_3), (c_2) and (d_3) do not occur because if $x=\infty$ then $y=\infty$. Furthermore the pair (d_2) and (b_3) do not occur because if $y=-\infty$ then $x=-\infty$.

If (a_1) occurs then observe the following. If the pair (a_2) and (a_3) occurs then $x\times z=\Phi\times\infty=y\times z$. If the pair ($b_2$) and ($b_3$) occurs then $x\times z=-\infty\times\infty=y\times z$. If the pair ($b_2$) and ($c_3$) occurs then $x\times z=-\infty\times\infty=-\infty<\infty=\infty\times\infty=y\times z$. If the pair ($b_2$) and ($d_3$) occurs then $x\times z=-\infty\times\infty=-\infty<\infty=y\times\infty=y\times z$. If the pair ($c_2$) and ($c_3$) occurs then $x\times z=\infty\times\infty=y\times z$. If the pair ($d_2$) and ($c_3$) occurs then $x\times z=x\times\infty=\infty=\infty\times\infty=y\times z$. If the pair ($d_2$) and ($d_3$) occurs then $x\times z=x\times\infty=\infty=y\times\infty=y\times z$.

If (b_1) occurs then observe the following. If the pair (a_2) and (a_3) occurs then $x\times z=\Phi\times z=y\times z$. If the pair ($b_2$) and ($b_3$) occurs then $x\times z=-\infty\times z=y\times z$. If the pair ($b_2$) and ($c_3$) occurs then $x\times z=-\infty\times z\leq\infty\times z=y\times z$. If the pair ($b_2$) and ($d_3$) occurs then, by hypothesis, $z\neq 0$, whence $x\times z=-\infty\times z=-\infty<y\times z$. If the pair ($c_2$) and ($c_3$) occurs then $x\times z=\infty\times z=y\times z$. If the pair ($d_2$) and ($c_3$) occurs, notice that in this case, by hypothesis, $z\neq 0$, whence $x\times z<\infty=\infty\times z=y\times z$. If the pair ($d_2$) and ($d_3$) occurs then the result follows from the real number order relation and real multiplication.

l) If $\infty\notin A$ and A is bounded above, in the real sense, then the result follows from the Supremum Axiom. Otherwise ∞ is the unique upper bound of A, whence $\infty=\sup A$.

□

4 Final Considerations

The transreal numbers have not been easily accepted [7]. We believe that one reason for the resistance to James Anderson's proposal is the fact that, in his presentation, the set of transreals is defined by $\mathbb{R}^T:=\mathbb{R}\cup\{-1/0,1/0,0/0\}$. By defining $\mathbb{R}^T$ in this way, Anderson presents a cyclical thought. He defines the transreals as being the reals joined to the elements $-1/0$, $1/0$ and $0/0$ and defines these elements as transreal numbers not real. That is, the objects $-1/0$, $1/0$ and $0/0$ are used to define themselves. Another reason for transreals appearing strange is that, in the new objects, the symbol "/" is undefined in context. Usually this symbol means division and a fraction with denominator zero has no sense in the set of real numbers (which is the set for which the arithmetical properties are already established). It is used as a symbol in the"old" representation to denote a "new" operation. That is, he uses the division symbol between real numbers to represent something not yet defined, the division between transreal numbers. It is worth mentioning that in our text, after a certain moment, we use also the symbol "/" to represent division between transreal numbers, but this is justified by Observation 8. We emphasize that this procedure is quite common in mathematics. Dedekind defines operations for addition and multiplication in the set of cuts and uses the same symbols for addition and for multiplication in the "old" arithmetic of the rationals and in the new operations because there is an isomorphism of ordered fields between a certain subset of cuts and the set of rational numbers. The same happens in many other cases, such as: the construction of the complexes from the reals, the construction of the hyperreals from the reals, the rationals from the integers and the integers from the naturals.

To solve the problem of cyclical thought, we use the concept of an equivalence relation. Notice that we want the fractions $-1/0$, $1/0$ and $0/0$ to be elements of the new set. Each fraction is determined by two real numbers, each one in a specific position. So the starting point was to think of each transreal number as an ordered pair of real numbers. The next step was to establish the criteria to consider two "fractions" (ordered pairs) as equivalent fractions. This justifies the relation created in Definition 1. And so we come to consider the quotient set $T/\sim$ and no just T. That is, a transreal number is not a pair of real numbers, but a certain class of ordered pairs of real numbers.

In Definition 4 we extend the arithmetical operations to the transreals. Note that the rules for obtaining the results of these operations are the same rules that customary practice dictates are used between fractions of real numbers, except for addition, whose definition was dismembered into two cases. Even so, addition can be obtained similarly to the well known practical rules for adding fractions of real numbers:

> *To sum two fractions, x and y, of real numbers . If x and y have the same denominator then copy the denominator into the result and add up the two numerators to give the numerator of the result. Otherwise create new fractions, with a common denominator, by multiplying the numerator and the denominator of x by the denominator of y and by multiplying the numerator and the denominator of y by the denominator of x then, as before, copy the new common denominator into the result and add up the numerators of the two new fractions to give the numerator of the result.*

In the transreal case:

> *To sum transreal numbers x and y. If $x = y$ then copy the second element into the result and add up the first elements to give the first element of the result. Otherwise multiply the first element and the second element of x by the second element of y, multiply the first element and the second element of y by the second element of x then, as before, copy the new second element into the result and add up the first elements of the two new pairs to give the first element of the result.*

We note that, of course, opposite does not means additive inverse and reciprocal does not mean multiplicative inverse. However, we stress that changing the meaning of operations, when it extends the concept of number, is a common occurrence. For example, for the natural numbers 3 and 6, the result of $6/3$ is the number of instalments, all equal to 3, whose sum is 6. This interpretation is meaningless when we operate on $3/6$. There is no number of instalments, all equal to 6, whose sum is 3. Of course, in the set of rational numbers, $3/6 = 0.5$, but this division no longer has the previous meaning. It makes no sense to say that the sum of 0.5 parts, all equal to 6, is equal to 3. We observe that if $[x, y] \in T/\sim$ then $-[x, y]$ does not mean the additive inverse of $[x, y]$, instead it means the image of $[x, y]$ in the function $[x, y] \longmapsto [-x, y]$. Likewise $[x, y]^{-1}$ does not mean the multiplicative inverse of $[x, y]$, it means the image of $[x, y]$ in the function $[x, y] \longmapsto \begin{cases} [y, x], & x \geq 0 \\ [-y, -x], & x < 0 \end{cases}$. Nevertheless we also observed that, when restricted to real numbers, the arithmetical operations defined on transreals coincide with the "old" operations of the reals.

It should be mentioned that the equivalence and arithmetic proposed here were motivated by the arithmetic of function limit theory. Note that if $k \in \mathbb{R}$ and $k > 0$ then $\lim\limits_{x \to 0^+} \dfrac{k}{x} = \infty$ [14]. This motivated us to define an equivalence relation so that if $k \in \mathbb{R}$ and $k > 0$ then $k/0 = \infty$. Among many other examples, we highlight that if $a \in \mathbb{R}$ and f and g are real functions such that $\lim\limits_{x \to a} f(x) = \infty$ and $\lim\limits_{x \to a} g(x) = \infty$ then $\lim\limits_{x \to a} (f(x) + g(x)) = \infty$. This motivated us to define an arithmetic such that $\infty + \infty = \infty$. We observe that we are proposing the enlargement of the number concept. And that, as already mentioned, this is not a new process in the development of mathematics. We are aware that the new set of numbers, $\mathbb{R}^T$, has some properties that appear somewhat unnatural in numbers. To cite one example, the distributive property does not hold for all transreal numbers, as seen in Example 14. However, at various moments in the extension of concepts, some properties are lost. Among many other examples we can point out that the set of the complex numbers is not an ordered field, as the reals are, the hyperreals do not have the Archimedean property that the reals have, the matrix product and product between Hamilton's quaternions are not commutative, unlike the reals, and in Cantor's transfinite arithmetic, addition is not commutative, unlike the reals.

5 Conclusion

The transreal numbers have an arithmetic which is closed over addition, subtraction, multiplication and division. The transreals have proved controversial and have not been readily accepted. We construct the set of transreal numbers from the set of real numbers and construct transreal arithmetic from real arithmetic. We show that the transreals contain the reals. We observe that, in the past, constructive proofs have ended controversies over the validity of new number systems.

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